

# A remark on $A + B$ and $A - A$ for compact sets in $\mathbb{R}^n$

By

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## Abstract

We prove in particular that if  $A \subset \mathbb{R}^n$  be a compact convex set, and  $B \subset \mathbb{R}^n$  be an arbitrary compact set then  $\mu(A - A) \ll \frac{\mu(A+B)^2}{\sqrt{n}\mu(A)}$ , provided that  $\mu(B) \geq \mu(A)$ .

A well-known Ruzsa triangle inequality states that for any finite sets of an abelian group we have

$$|A - B| \leq \frac{|A + C||C + B|}{|C|},$$

in particular if  $B = A$  and  $C = B$ , then

$$|A - A| \leq \frac{|A + B|^2}{|B|}.$$

The aim of this note is to prove a sharp up to a dimension-independent constant form of the above inequality for a compact convex set  $A \subset \mathbb{R}^n$ , and an arbitrary compact set  $B \subset \mathbb{R}^n$ , provided that  $\mu(A) \geq \mu(B)$ .

For a set  $A \subset \mathbb{R}^n$  and  $x \in A - A$  put

$$A_x = A \cap (A - x).$$

Our main tool is the following lemma proved in [4] (Lemma 5). We recall its proof as it is very simple.

**Lemma 1** *Let  $A, B \subset \mathbb{R}^n$  be compact sets. Then*

$$\int_{A-A} \mu(A_x + B) dx \leq \mu(A + B)^2. \quad (1)$$

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**Proof.** We apply a well-known Koester-Katz transform: if  $x \in A - A$  then

$$A_x + B \subseteq (A + B)_x.$$

Therefore, we have

$$\int_{A-A} \mu(A_x + B) dx \leq \int_{A+B-A-B} \mu((A+B)_x) dx = \mu(A+B)^2,$$

and the assertion follows.  $\square$

We also need a lower bound for the size of  $A_x$  for a convex set  $A$ , see [5] section 3. We also give the proof for the sake of completeness.

**Lemma 2** *Let  $A \subset \mathbb{R}^n$  be a compact convex set and  $r \in [0, 1]$  be any real number. Then for all  $x \in r(A - A)$  the following holds*

$$\mu(A_x) \geq (1-r)^n \mu(A). \quad (2)$$

**Proof.** Write  $x = ra_1 - ra_2$ , where  $a_1, a_2 \in A$  and let  $a \in A$  be any element. By convexity  $(1-r)a + ra_1 \in A$  and  $(1-r)a + ra_1 = (1-r)a + ra_2 + x \in A + x$ . Thus  $(1-r)A + ra_1 \subseteq A \cap (A + x)$  and the result follows.  $\square$

Finally, we recall the Brunn-Minkowski inequality, see [5] section 3.

**Theorem 3** *Let  $A, B \subset \mathbb{R}^n$  be non-empty compact sets. Then*

$$\mu(A + B)^{1/n} \geq \mu(A)^{1/n} + \mu(B)^{1/n}.$$

Now we can formulate our main result.

**Theorem 4** *Let  $A \subset \mathbb{R}^n$  be a compact convex set, and  $B \subset \mathbb{R}^n$  be an arbitrary compact set. Then*

$$(1 + \omega + \dots + \omega^{[\sqrt{n}]}) \mu(B)^{1-1/n} \mu(A)^{1/n} \mu(A - A) \ll \mu(A + B)^2, \quad (3)$$

where  $\omega = (\mu(A)/\mu(B))^{1/n}$ . In particular, if  $\mu(A) \geq \mu(B)$  then

$$\mu(A - A) \ll \frac{\mu(A + B)^2}{\sqrt{n} \mu(A)^{1/n} \mu(B)^{1-1/n}}, \quad (4)$$

and if  $\mu(B) \geq \mu(A)$  then

$$\mu(A - A) \ll \frac{\mu(A + B)^2}{\sqrt{n} \mu(A)}. \quad (5)$$

**Proof.** Let  $\alpha = \mu(B)/\mu(A)$ . Applying (1) and the Brunn-Minkowski inequality, we get

$$\begin{aligned}\mu^2(A+B) &\geq \int_{A-A} \mu(B+A_x) dx \geq \int_{A-A} \left( \mu(B)^{1/n} + \mu(A_x)^{1/n} \right)^n dx \\ &= \alpha \sum_{k=0}^n \binom{n}{k} \int_{A-A} \alpha^{-k/n} \mu(A)^{(n-k)/n} \mu(A_x)^{k/n} dx.\end{aligned}$$

To estimate the size of  $A_x$  we use Lemma 2. After integration by parts, we obtain

$$\begin{aligned}\mu^2(A+B) &\geq \mu(B) \sum_{k=0}^n \binom{n}{k} k \alpha^{-k/n} \int_0^1 (1-r)^{k-1} \mu(r(A-A)) dr \\ &= \mu(B) \mu(A-A) \sum_{k=1}^n \binom{n}{k} k \alpha^{-k/n} \int_0^1 (1-r)^{k-1} r^n dr \\ &= \mu(B) \mu(A-A) \sum_{k=1}^n \binom{n}{k} k \alpha^{-k/n} \mathcal{B}(k, n+1),\end{aligned}$$

where  $\mathcal{B}(\cdot, \cdot)$  is the beta function. Thus

$$\mu^2(A+B) \geq \mu(B) \mu(A-A) \sum_{k=1}^n \alpha^{-k/n} \frac{(n!)^2}{(n-k)!(n+k)!} := \mu(B) \mu(A-A) \times \sigma.$$

One can calculate the last sum  $\sigma$  using the gamma function or hypergeometric series, but we use a rather crude estimate. Put  $\Delta = \lfloor \sqrt{n} \rfloor + 1$ , then

$$\sigma = \sum_{k=1}^n \alpha^{-k/n} \prod_{j=1}^{k-1} \left(1 - \frac{j}{n}\right) \prod_{j=1}^k \left(1 + \frac{j}{n}\right)^{-1} = \sum_{k=1}^n \alpha^{-k/n} \left(1 + \frac{k}{n}\right)^{-1} \prod_{j=1}^{k-1} \left(1 - \frac{2j}{n+j}\right).$$

Using inequalities  $\ln(1-x) \geq -2x$  for  $0 \leq x \leq 0.5$  and  $k \leq n$ , we obtain

$$\sigma \geq \frac{1}{2} \sum_{k=1}^{\Delta} \alpha^{-k/n} \exp \left( - \sum_{j=1}^{k-1} \frac{4j}{n+j} \right) \geq \frac{1}{2} \sum_{k=1}^{\Delta} \alpha^{-k/n} \exp \left( - \frac{2k^2}{n} \right) \gg \sum_{k=1}^{\Delta} \omega^k.$$

This gives us (3). To see (4) it is enough to observe that if  $\mu(A) \geq \mu(B)$  then  $\sum_{k=1}^{\Delta} \omega^k \geq \sqrt{n}$ . To get (5) take any subset  $B'$  of  $B$  such that  $\mu(B') = \mu(A)$  and apply (4), then

$$\mu(A-A) \ll \frac{\mu(A+B')^2}{\sqrt{n}\mu(A)} \leq \frac{\mu(A+B)^2}{\sqrt{n}\mu(A)}.$$

This completes the proof.  $\square$

**Remark 5** Estimate (4) is tight, see paper [2] or book [3], discussion after Corollary 8.3. Indeed, consider  $n$ -dimensional simplex

$$A = A_L = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_j \geq 0, \sum_{j=1}^n x_j \leq L\},$$

where  $L$  is a parameter. Then  $\mu(A + A) = 2^n \mu(A)$  and  $\mu(A - A) = \binom{2n}{n} \mu(A)$  (to obtain the last formula one can count to number of integer points in  $A$ , say, and approximate  $\mu(A - A)$  by

$$\sum_{a+b+c=n} \binom{n}{a,b,c} \binom{L}{a} \binom{L}{b} \sim \frac{L^n}{n!} \sum_{m=0}^n \binom{n}{m}^2 = \frac{L^n}{n!} \binom{2n}{n} = \mu(A) \binom{2n}{n},$$

see [1]. Here  $a, b, c$  the number of possibilities for the positive, negative and zero coordinates in  $A - A$ , correspondingly). Hence

$$\mu(A - A) \gg \frac{\mu(A + A)^2}{\sqrt{n} \mu(A)}.$$

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